PAM3012
Digital Image Processing for Radiographers

The Fourier Transform & The Frequency Domain

In this lecture

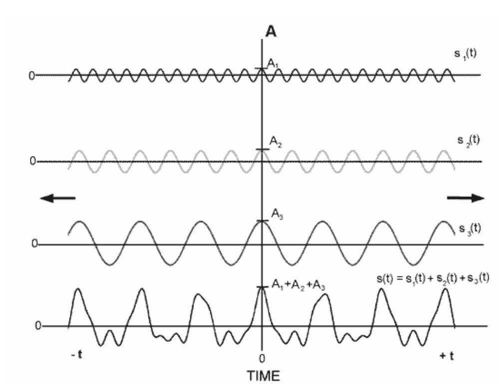
- ★Frequency domain
- ★1D Fourier transform and it's inverse
- ★2D Fourier transform and it's inverse
- ★Properties of the Fourier Transform

Frequency Domain

What is the frequency domain & where does it fit into image processing?

Background

- *Fourier Series:*
  - Any periodic function can be expressed as a sum of sines and/or cosines of different frequencies and amplitudes
- *Fourier Transform:*
  - Non-periodic functions can be expressed as an integral of sines and/or cosines multiplied by weighting factors

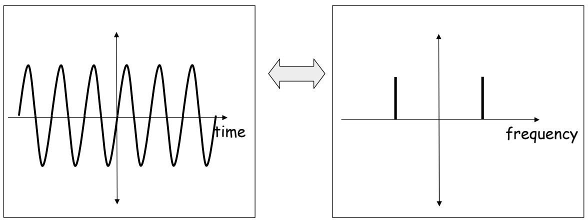


Background

- A function represented as a *Fourier series* or *transform* can be recovered completely via an inverse process, with no loss of information
- Allows us to work in the *Fourier domain* and then return to the original domain

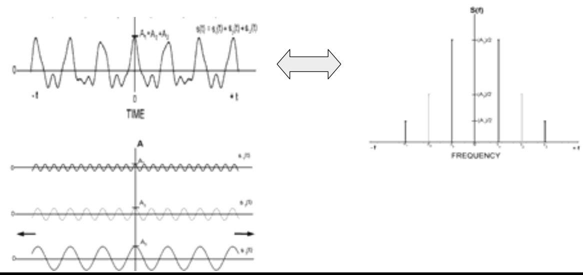
Background

Time Domain | Fourier Domain



Background

Time Domain | Fourier Domain?



Fourier Transform & Frequency Domain

- 1D Fourier Transform
- 2D Fourier Transform
- Discrete Formulation
- Properties

1-Dimensional Fourier Transform

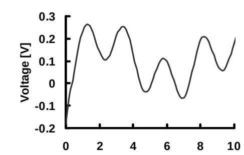
Fourier Transform & Frequency Domain

- 1D Function *f(x)*

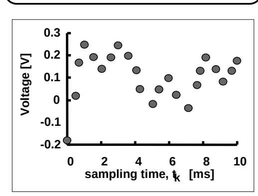
$$f(x) \underset{\text{Inverse Fourier Transform}}{\overset{\text{Fourier Transform}}{\rightleftarrows}} F(u)$$

Discrete Fourier Transform

Continuous function V of *continuous* variable t.



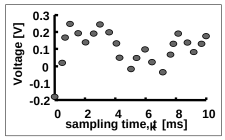
Discrete function V_k of *discrete* sampling variable t_k.



- Digital signals are discrete
  Discrete Fourier Transform (DFT)

Discrete Fourier Transform

- Discrete function: *f(x)*
- *x = 0, 1, 2, … M-1*

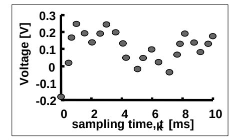


$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-\frac{j2\pi ux}{M}} \quad \text{(Equation 1)}$$

For u = 0, 1, 2, … M-1

Inverse Discrete Fourier Transform

- Discrete function: *F(u)*
- *u = 0, 1, 2, … M-1*



$$f(x) = \sum_{u=0}^{M-1} F(u)\, e^{\frac{j2\pi ux}{M}} \quad \text{(Equation 2)}$$

For x = 0, 1, 2, … M-1

Discrete Fourier Transform

- *F(u)* & *f(x)* are known as a *Fourier Transform Pair*

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-\frac{j2\pi ux}{M}} \quad \text{(Equation 1)}$$

$$f(x) = \sum_{u=0}^{M-1} F(u)\, e^{\frac{j2\pi ux}{M}} \quad \text{(Equation 2)}$$

Discrete Fourier Transform

Computing DFT $F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-\frac{j2\pi ux}{M}}$

- Substitute u=0 & sum over all values of x
- Repeat for M values of u
- Total of M^2 summations & multiplications
- F(u) has same number of components as f(x) & is a discrete quantity

Discrete Fourier Transform

- Unlike continuous Fourier transform DFT and it's inverse always exists
- Can be shown by substituting Equation 1 into Equation 2

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-\frac{j2\pi ux}{M}} \quad \text{(Equation 1)}$$

$$f(x) = \sum_{u=0}^{M-1} F(u)\, e^{\frac{j2\pi ux}{M}} \quad \text{(Equation 2)}$$

Frequency Domain

- Euler's Formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

- Substitute into equation 1

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos\left(\frac{2\pi ux}{M}\right) - j\sin\left(\frac{2\pi ux}{M}\right)\right]$$

For u = 0, 1, 2, … M-1

Frequency Domain

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos\left(\frac{2\pi ux}{M}\right) - j\sin\left(\frac{2\pi ux}{M}\right)\right]$$

- Each term of Fourier Transform (i.e. value of F(u) for each value of u) is composed of sum of all values of *f(x)*
- Each value of *f(x)* is multiplied by sine & cosine of various frequencies

Frequency Domain

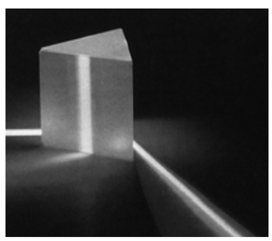
$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos\left(\frac{2\pi ux}{M}\right) - j\sin\left(\frac{2\pi ux}{M}\right)\right]$$

- Domain (u) over which values of F(u) range is appropriately called the frequency domain
- Each of the M terms is called the frequency component

Frequency Domain

Analogy: Prism

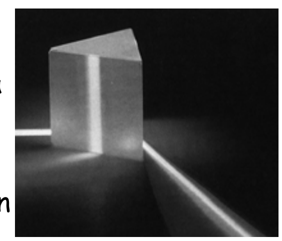
- Physical device that separates white light into its constituent colours.
- Each colour depends on it wavelength or frequency



Frequency Domain

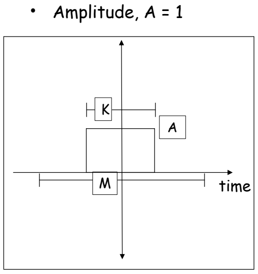
Analogy: Prism

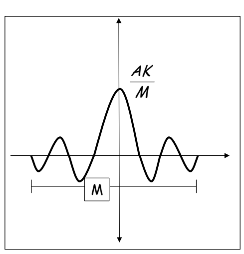
- Fourier Transform is a 'mathematical prism'
- Allows us to characterise a function by its frequency content



1-Dimensional DFT

- Discrete function *f(x)*
- Total of M data points
- K = 8 points
- Amplitude, A = 1

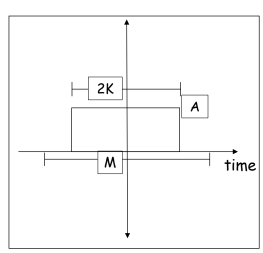


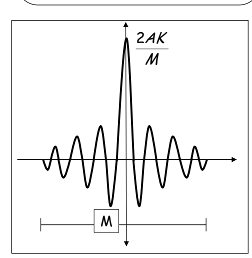


1-Dimensional DFT

- Discrete function *f(x)*
- Total of M data points
- K = 16 points
- Amplitude, A = 1

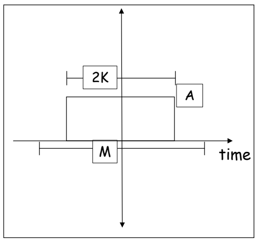
Note:
1. Height Doubled
2. Intercepts Double
3. Reciprocal Nature of FT

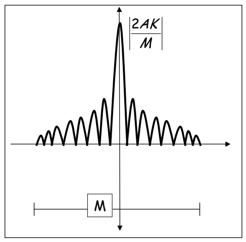




1-Dimensional DFT

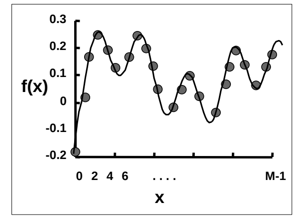
- When dealing with images only interested in magnitude of signal infrequency domain
- Magnitude of *f(x)* = |*f(x)*|





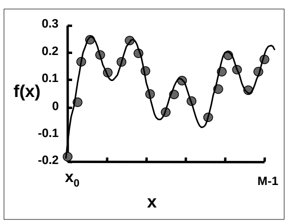
Discrete Fourier Transform

- M discrete data points sampled from a continuous signal
- f(x) for x = 0, 1 … M-1
- Samples not necessarily taken at integer values



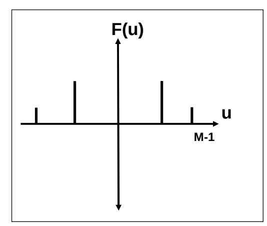
Discrete Fourier Transform

- Usually represented by x_0 denoting 1^{st} point
  - Value of sampled function $f(x_0)$
- Next sample taken at fixed interval from x_0
  - Value of sampled function $f(x_0 + \Delta x)$
- k^{th} sample value $f(x_0 + k\Delta x)$
- Final sample value $f(x_0 + [M-1]\Delta x)$



Discrete Fourier Transform

- F(u) has similar properties, but sequence always starts at true zero frequency
- u = 0, Δu, 2Δu … [M-1] Δu
- Δx and Δu are inversely related



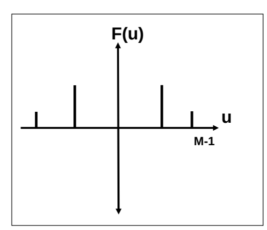
$$\Delta u = \frac{1}{M\Delta x}$$

Example

- A continuous signal is sampled with a 1 second interval between data points. If total of 1000 data points are sampled what is the increments size of the signal in the frequency domain?

Discrete Fourier Transform

- What is the maximum frequency that is expressed by the FT?



$$u_{max} = \frac{1}{2\Delta x}$$

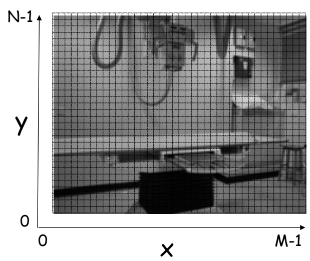
Example

- A continuous signal is sampled with a 1 second interval between data points. If total of 1000 data points are sampled what is the maximum frequency appearing in the frequency domain?

2-Dimensional Fourier Transform

Digital Image

- Digital Image can be described by a discrete 2D function
- *x, y* and *f(x, y)* are all finite and discrete
- *f(x, y)* is the gray level in each pixel
- Define number of x pixels M & number of y pixels N



Fourier Transform & Frequency Domain

- 2D Function *f(x, y)*

$$f(x, y) \xrightarrow{\text{Fourier Transform}} F(u, v)$$

$$f(x, y) \xleftarrow{\text{Inverse Fourier Transform}} F(u, v)$$

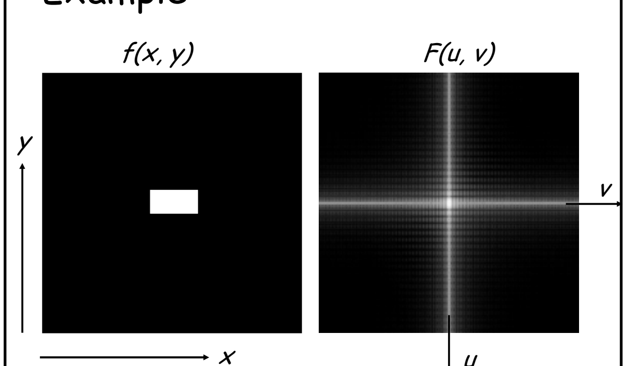
2D Discrete Fourier Transform

- Equations 1 & 2 become

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

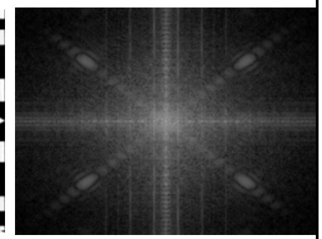
Example



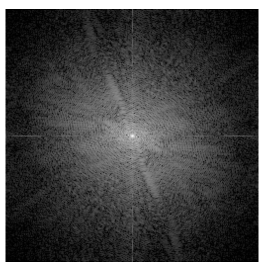
Frequency Domain

- Values of F(u, v) contain all values of f(x,y) modified by exponential
- Impossible to make direct associations between specific components of image and its FT
- General statement can be made
  - Where u = v = zero:
    - Average gray-level of image
  - Frequency (rate of change):
    - Patterns of intensity variations

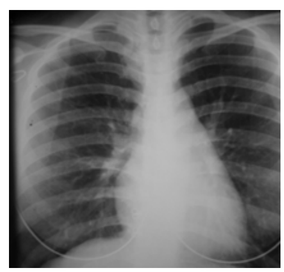
Example



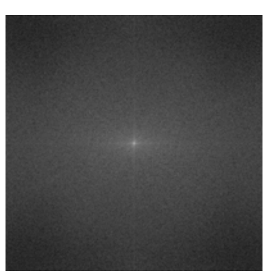
Example

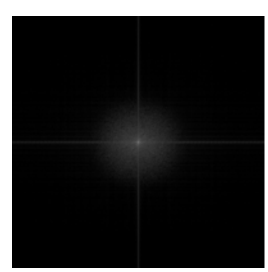
Example




Example

Example

Summary

★Frequency domain

★1D Fourier transform and it's inverse

★2D Fourier transform and it's inverse

★Properties of the Fourier Transform